THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MMAT5210 Discrete Mathematics 2017-2018
Suggested Solution to Assignment 1

1. Define a relation $\sim$ on $\mathbb{R}^2$ such that $(x_1, y_1) \sim (x_2, y_2)$ if and only if $x_1^2 + y_1^2 = x_2^2 + y_2^2$.

   (a) Show that the relation $\sim$ is an equivalence relation.

   (b) What are the elements of the equivalence class $[(3, 4)]$?

   (c) Describe the elements of $\mathbb{R}^2/\sim$.

   Ans:

   (a) Obviously, above relation satisfies the reflexive, symmetric and transitive properties. Hence, it is an equivalence relation.

   (b) By definition, an element (x, y) in the class $[(3, 4)]$ satisfies equation $x^2 + y^2 = 25$. Therefore, the point which lies in the circle $x^2 + y^2 = 25$ is element of the equivalence class $[(3, 4)]$.

   (c) By definition, any two elements lie in the same class if and only if they lie in the same circle in $\mathbb{R}^2$. Therefore, $\mathbb{R}^2/\sim = [0, +\infty)$.

2. Let $\mathbb{R}[x]$ be the set of all polynomials with real coefficients.

   Define a relation $\sim$ on $\mathbb{R}[x]$ such that $P(x) \sim Q(x)$ if and only if $P(x) - Q(x)$ is divisible by $x^2 + x + 1$.

   (a) Show that the relation $\sim$ is an equivalence relation.

   (b) Describe the elements of $\mathbb{R}[x]/\sim$.

   Ans:

   (a) Obviously, above relation satisfies the reflexive and symmetric properties. To see the transitive property, note that if $P(x)-Q(x)$ and $Q(X)-R(x)$ are divisible by x^2+x+1, then $P(x)-R(x)$ is also divisible by $x^2 + x + 1$.

   (b) Since every $P(x)$ can be written as $P(x) = K(x)(x^2+x+1)+ax+b$ and $ax+b$ is not divisible by $x^2 + x + 1$, $\mathbb{R}[x]/\sim = \{[ax + b] | a, b \in \mathbb{R}\}$.

3. Construct a concrete bijective function from $\mathbb{Z}^+$ to $\mathbb{Z}$.

   Ans: Define function

$$f(n) = \begin{cases} -\frac{n+1}{2} & \text{if } n \text{ is odd} \\ \frac{n}{2} & \text{if } n \text{ is even} \end{cases}$$

   To see that f is injective, if there are n_1 and n_2 such that $f(n_1) = f(n_2)$. By definition, both of n_1 and n_2 are even if $f(n_1)$ is positive and both of n_1 and n_2 are odd if $f(n_1)$ is negative. According to the definition, we know that $n_1 = n_2$.

   Obviously, f is surjective.

4. Show that $\log(n^2+1)$ is $O(\log n)$.

Ans: In that $2\log n = \log n^2 \leq \log(n^2+1) \leq \log n^3 = 3\log n$, we have $\log(n^2+1) = O(\log n)$.

5. (a) Show that $n \log n$ is $O(\log n!)$.

   Ans: In that $n! = n \times \cdots \times 1 \geq (\frac{n}{2})^{\frac{n}{2}}$, we have

$$\log n! \geq \frac{n}{2}(\log n - \log 2).$$

   Therefore, $10 \log n! \geq n \log n$ for $n \geq 2$.

   (b) Is it true that $n \log n$ is $\Theta(\log n!)$?

   Ans: Yes, in that $n! \leq n^n$, so $\log n! \leq n \log n$.

6. Let $\alpha > \beta > 1$. Show that β^n is $O(\alpha^n)$, but α^n is not $O(\beta^n)$.

Ans: By assumption, we know that $0 < \frac{\beta}{\alpha} < 1$ and hence $\lim_{n\to\infty} \frac{\beta^n}{\alpha^n} = 0$. Therefore β^n is $O(\alpha^n)$ and α^n is not $O(\beta^n)$.

7. Let k be a positive integer.

   (a) Show that $1^k + 2^k + \cdots + n^k$ is $O(n^{k+1})$.

   Ans: It is worth noting that $1^k + 2^k + \cdots + n^k \leq n^k + n^k + \cdots + n^k = n^{k+1}$, therefore, $1^k + 2^k + \cdots + n^k$ is $O(n^{k+1})$..

   (b) Is it true that $1^k + 2^k + \cdots + n^k$ is $\Theta(n^{k+1})$?

   Ans: Yes, this is true. By definition of the Riemann integral we have:

$$\lim_{n\to\infty} \frac{1^k + 2^k + \cdots + n^k}{n^{k+1}} = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} (\frac{i}{n})^k = \int_0^1 x^k dx = \frac{1}{k+1}.$$

   Therefore, n^{k+1} is $O(1^k + 2^k + \cdots + n^k)$.

8. (a) Show that for any positive integer $n \geq 2$,

$$\sum_{j=2}^{n} \frac{1}{j} < \int_1^n \frac{1}{x}\, dx.$$

   Ans: In that $\frac{1}{x}$ is strictly decreasing, we have

$$\int_1^n \frac{1}{x} dx = \sum_{i=2}^{n} \int_{i-1}^{i} \frac{1}{x} dx > \sum_{i=2}^{n} \int_{i-1}^{i} \frac{1}{i} dx = \sum_{i=2}^{n} \frac{1}{i}.$$

   (b) Let H_n be the n-th harmonic number

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

   By using the result in (a), show that H_n is $O(\log(n))$

Ans: $H_n = 1 + \sum_{j=2}^{n} \frac{1}{j} < 1 + \int_1^n \frac{1}{x}\, dx = 1 + \log n \leq C \log n$. Hence, H_n is $O(\log(n))$.